

# Particle interferometry, binary sources and oscillations in two-particle correlations

T. Csörgő  
MTA KFKI RMKI  
H - 1525 Budapest 114, POB 49  
Hungary

The basics and the formalism of Bose-Einstein correlations is briefly reviewed. The invariant Buda-Lund form is summarized. Tools are presented that can be utilized in a model-independent search for non-Gaussian structures in the two-particle Bose-Einstein correlation functions. The binary source formalism of particle interferometry is presented and related to oscillations in the two-particle Bose-Einstein and Fermi-Dirac correlation functions. The frequency of the observed oscillations in the NA49 two-proton correlation function in Pb+Pb collisions at CERN SPS energies is explained with the help of the reconstructed space-time picture of particle production and the binary nature of the proton source in this reaction.

## 1. Introduction

Essentially, intensity correlations appear due to the Bose-Einstein or Fermi-Dirac symmetrization of the two-particle final states of identical bosons or fermions, in short, due to quantum statistics. Intensity correlations were discovered for the first time in radio astronomy by R. Hanbury Brown and R. Q. Twiss, [1,2] and were utilized to determine the angular diameter of main sequence stars, the HBT effect. In particle physics, the enhancement of the production of identical pions at small angular separations was discovered by Goldhaber, Goldhaber, Lee and Pais (GGLP) in refs. [3,4].

In this contribution, some comments are made on the frequently invoked fully thermal and fully chaotic limiting cases and it is shown why the Andersson-Hofmann model corresponds to neither of these cases. Certain subtle aspects of two-particle Bose-Einstein and Fermi-Dirac correlations are highlighted, that can be utilized in experimental searches for new, non-Gaussian structures in the two-particle quantum statistical correlation functions. Some of the material discussed here is described in greater details in the recent review paper [5]. Various complementary aspects of the field were reviewed recently in refs. [6–12].

### 1.1. Basics of intensity correlations

The simplest derivation of the HBT/GGLP/Bose-Einstein correlation effect is as follows: suppose that a particle pair is observed, one with momentum  $k_1$  the other with momentum  $k_2$ . The amplitude of pair emission has to be symmetrized over the unobservable variables, in particular over the points of emissions  $x_1$  and  $x_2$ . If Coulomb, strong or other final state interactions can be neglected, the amplitude of such a final state is proportional to

$$A_{12} \propto \frac{1}{\sqrt{2}} [e^{ik_1x_1+ik_2x_2} \pm e^{ik_1x_2+ik_2x_1}], \quad (1)$$

where + sign stands for bosons, – for fermions. If the particles are emitted in an incoherent manner, from a non-expanding source, the observable two-particle spectrum is proportional to

$$N_2(k_1, k_2) \propto \int dx_1 \rho(x_1) \int dx_2 \rho(x_2) |A_{12}|^2 \quad (2)$$

and the resulting two-particle intensity correlation function is

$$C_2(k_1, k_2) = \frac{N_2(k_1, k_2)}{N_1(k_1)N_1(k_2)} = 1 \pm |\tilde{\rho}(q)|^2, \quad (3)$$

that carries information about the Fourier-transformed space-time distribution of the par-

ticle emission

$$\tilde{\rho}(q) = \int dx e^{iqx} \rho(x). \quad (4)$$

as a function of the relative four-momentum  $q \equiv q_{12} = k_1 - k_2$ .

Although this derivation is over-simplified, the above result can be translated to the more generally valid identity:

$$\langle a_1^\dagger a_2^\dagger a_2 a_1 \rangle = \langle a_1^\dagger a_1 \rangle \langle a_2^\dagger a_2 \rangle \pm \langle a_1^\dagger a_2 \rangle \langle a_2^\dagger a_1 \rangle, \quad (5)$$

where  $a_i^\dagger$  and  $a_i$  are the creation and annihilation operators for identical particles with 4-momentum  $k_i$ , ( $i = 1, 2$ ) and  $\langle O \rangle = \text{Tr} \rho O$  stands for the expectation value of the operator  $O$  in a system characterized by a density matrix  $\rho$ .

A more sophisticated derivation of this and the corresponding  $n$ -particle Bose-Einstein correlation functions is reviewed e.g. in ref. [5]. Here we highlight only the essential properties of various statistical features of the quantum statistical correlations, hence the mathematical complications will be limited to the minimal level.

As compared to the idealized case when quantum-statistical correlations are negligible (or neglected), Bose-Einstein or Fermi-Dirac correlations modify the momentum distribution of the hadron pairs in the final state by a weight factor

$$\langle a_1^\dagger a_2^\dagger a_2 a_1 \rangle = \langle a_1^\dagger a_1 \rangle \langle a_2^\dagger a_2 \rangle \{1 \pm |\langle e^{iq_{12}x} \rangle|^2\}. \quad (6)$$

Note that eq. (5) is valid for chaotic (for example, locally thermalized) systems and it is in a very good agreement with the detailed experimental and theoretical investigations of quantum statistical correlation functions in high energy heavy ion collisions.

However, eq. (5) is not valid in case of a coherent particle emission. In case of a brehmstrahlung - like or laser - like coherent radiation, eq. (5) is replaced by

$$\langle a_1^\dagger a_2^\dagger a_2 a_1 \rangle = \langle a_1^\dagger a_1 \rangle \langle a_2^\dagger a_2 \rangle. \quad (7)$$

Thus the non-trivial quantum statistical intensity correlations appear in chaotic, thermal like sources. They correspond to the second term of eq. (5). Second order optical coherence is defined

by the vanishing value of these kind of exchange terms, compare eq. (7) with eq. (5).

Thus the applicability of the starting point of many detailed derivations, eq. (5) is limited to chaotic, thermal-like sources. If some degree of coherence is preserved during particle production, eqs. (5,6) become invalid. This is the case of not only in case of a coherent, laser-like radiation but also in case of the Andersson-Hofmann model of Bose-Einstein correlations in  $e^+e^-$  collisions.

Already at this basic level, one can realize the uniqueness of the Andersson-Hofmann model of a two-particle Bose-Einstein correlations in  $e^+e^-$  annihilation.

## 2. The Andersson-Hofmann model

The hadronic production in  $e^+e^-$  annihilations is usually considered to be a basically coherent process and therefore no Bose-Einstein effect was expected in these reactions 20 years ago. It was also thought that the hadronic reactions should be of a more chaotic nature giving rise to a sizeable effect. It was even argued that the strong ordering in rapidity, preventing neighbouring  $\pi^-\pi^-$  or  $\pi^+\pi^+$  pairs, would drastically reduce the effect [13]. Therefore it was a surprise when G. Goldhaber at the Lisbon Conference in 1981 [14] presented data which showed that correlations between identical particles in  $e^+e^-$  annihilations were very similar in size and shape to those seen in hadronic reactions, see the review paper ref. [6] for further details.

The Bose-Einstein correlation effect, *a priori* unexpected for a coherent process, has been given an explanation within the Lund string model by B. Andersson and W. Hofmann [15]. The space-time structure of an  $e^+e^-$  annihilation is shown for the Lund string model [16] in Figure 1. The probability for a particular final state is given by the Lund area law

$$\text{Prob} \sim \text{phasespace} \cdot \exp(-bA), \quad (8)$$

where  $A$  is the space-time area spanned by the string before it breaks and  $b$  is a parameter. The classical string action is given by  $S = \kappa A$ , where  $\kappa$  is the string tension. It is natural to interpret the result in eq. (8) as resulting from an imaginary

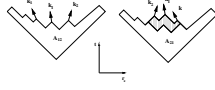


Figure 1. Andersson - Hofmann interpretation of Bose-Einstein correlations in the Lund string model.  $A_{12}$  denotes the space-time area of a colour field enclosed by the quark loop in  $e^+e^-$  annihilation. Two particles 1 and 2 are separated by the intermediate system  $I$ . When the particles 1 and 2 are identical, the configuration in the left side is indistinguishable from that of the right side, and their amplitudes for production must be added. The probability of production will depend on the difference in area  $\Delta A = A_{12} - A_{21}$ , shown as the hatched area.

part of the action such that

$$S = \xi A, \quad (9)$$

$$\text{Re}\xi = \kappa, \quad (10)$$

$$\text{Im}\xi = b/2, \quad (11)$$

and an amplitude  $M$  given by

$$M \sim \exp(iS), \quad (12)$$

which implies

$$\text{Prob} \sim |M|^2 \sim \exp(-bA). \quad (13)$$

Final states with two identical particles are indistinguishable and can be obtained in different ways. Suppose that the two particles indicated as 1 and 2 on Fig. 1 are identical, then the hadron state in the left panel can be considered as being the same as that in the right panel (where 1 and 2 are interchanged). The amplitude should, for bosons, be the sum of two terms

$$M \sim M_{12} + M_{21} = \exp[i\xi A_{12}] + \exp[i\xi A_{21}] \quad (14)$$

where  $A_{12}$  and  $A_{21}$  are the two string areas, giving a probability proportional to

$$|M|^2 \sim [\exp(-bA_{12}) + \exp(-bA_{21})] \times \left[1 + \frac{\cos(\kappa\Delta A)}{\cosh(b\Delta A/2)}\right] \quad (15)$$

with  $\Delta A \equiv A_{12} - A_{21}$ . The magnitudes of  $\kappa$  and  $b$  are known from phenomenological studies. The energy per unit length of the string is given by  $\kappa \approx 1$  GeV/fm, and  $b$  describes the breaking of the string at a constant rate per unit area,  $b/\kappa^2 \approx 0.7$  GeV $^{-2}$  [16]. The difference in space-time area  $\Delta A$  is marked as the hatched area in Fig. 1. It can be expressed by the  $(t, r_z)$  components  $(E, k)$  of the four-momenta of the two identical particles 1 and 2, and the intermediate system  $I$ :

$$\Delta A = [E_2 k_1 - E_1 k_2 + E_I(k_1 - k_2) - k_I(E_1 - E_2)]/\kappa^2$$

To take into account also the component transverse to the string a small additional term is needed. The change in area  $\Delta A$  is Lorentz invariant to boosts along the string direction and is furthermore approximately proportional to  $Q = \sqrt{-(k_1 - k_2)^2}$ .

The interference pattern between the amplitudes will be dominated by the phase change of  $\Delta\Phi = \kappa\Delta A$ . It leads to a Bose-Einstein correlation which, as a function of the four-momentum transfer, reproduces the data well but shows a steeper dependence at small  $Q$  than a Gaussian function. A comparison to TPC data confirmed the existence of such a steeper than Gaussian dependence on  $Q$ , although the statistics at the small  $Q$ -values did not allow a firm conclusion [6,17].

An essential feature of the Andersson - Hofmann model is that

$$\begin{aligned} \langle a_1^\dagger a_2^\dagger a_2 a_1 \rangle &= \langle |M_{12} + M_{21}|^2 \rangle \\ &\neq \langle M_{11} \rangle \langle M_{22} \rangle + |\langle M_{12} \rangle|^2 \end{aligned} \quad (16)$$

and

$$\begin{aligned} \langle a_1^\dagger a_2^\dagger a_2 a_1 \rangle &= \langle |M_{12} + M_{21}|^2 \rangle \\ &\neq \langle M_{11} \rangle \langle M_{22} \rangle \end{aligned} \quad (17)$$

i.e. the source is neither chaotic, nor fully coherent. In case of a well defined intermediate system  $I$ , the phase difference between the two amplitudes is also well defined; however,  $I$  is randomly varying from event to event, that leads to a variation of the phases and the onset of a chaotic like behaviour. As the phase difference  $\propto \Delta A$  is, however, not a uniformly distributed random variable

in the Lund model, a residual phase correlation survives in the two-particle Bose-Einstein correlation function, that cannot be obtained otherwise, neither in fully chaotic, nor in fully coherent systems.

In the subsequent parts, we consider only fully chaotic systems, relevant for the study of hadron-hadron, hadron-nucleus and nucleus-nucleus interactions at high energies.

### 3. Buda-Lund particle interferometry

The  $n$ -particle Bose-Einstein correlation function is defined as the ratio of the  $n$ -particle invariant momentum momentum distribution divided by an  $n$ -fold product of the single-particle invariant momentum distributions. Hence these correlation functions are boost-invariant.

The invariant Buda-Lund parameterization (or BL in short) deals with a boost-invariant, multi-dimensional characterization of the building blocks  $\langle a_{\mathbf{k}_1}^\dagger a_{\mathbf{k}_2} \rangle$  of arbitrary high order Bose-Einstein correlation functions. The BL parameterization was developed by the Budapest-Lund collaboration in refs. [18,19].

The essential part of the BL is an invariant decomposition of the relative four-momentum  $q$  in the  $\langle \exp(iqx) \rangle$  factor of eq. (6) into a temporal, a longitudinal and two transverse relative momentum components. This decomposition is obtained with the help of a time-like vector in the coordinate space, that characterizes the center of particle emission in space-time, see Fig. 2.

Although the BL parameterization was introduced in ref. [18] for high energy heavy ion reactions, it can be used for other physical situations as well, where a dominant direction of an approximate boost-invariant expansion of the particle emitting source can be identified and taken as the longitudinal direction  $r_z$ . For example, such a direction is the thrust axis of single jets or of back-to-back two-jet events in case of high energy particle physics. For longitudinally almost boost-invariant systems, it is advantageous to introduce the boost invariant variable  $\tau$  and the space-time rapidity  $\eta$ ,

$$\tau = \sqrt{t^2 - r_z^2}, \quad (18)$$

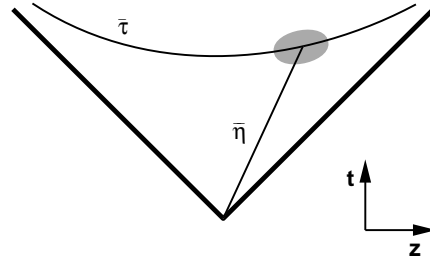


Figure 2. Space-time picture of particle emission for a given fixed mean momentum of the pair. The mean value of the proper-time and the space-time rapidity distributions is denoted by  $\bar{\tau}$  and  $\bar{\eta}$ . As the rapidity of the produced particles changes from the target rapidity to the projectile rapidity the  $[\bar{\tau}(y), \bar{\eta}(y)]$  variables scan the surface of mean particle production in the  $(t, r_z)$  plane.

$$\eta = 0.5 \log [(t + r_z)/(t - r_z)]. \quad (19)$$

Similarly, in momentum space one introduces the transverse mass  $m_t$  and the rapidity  $y$  as

$$m_t = \sqrt{E^2 - p_z^2}, \quad (20)$$

$$y = 0.5 \log [(E + p_z)/(E - p_z)]. \quad (21)$$

The source of particles is characterized in the boost invariant variables  $\tau$ ,  $m_t$  and  $\eta - y$ . For systems that are only approximately boost-invariant, the emission function may also depend on the deviation from mid-rapidity,  $y_0$ . The scale on which the approximate boost-invariance breaks down is denoted by  $\Delta\eta$ , a parameter that is related to the width of the rapidity distribution.

The correlation function is defined with the help of the Wigner-function formalism, the intercept parameter  $\lambda_*$  is introduced in the core-halo picture see refs. [18,20,5] for further details. In the following, we evaluate the Fourier-transformed Wigner functions, that provide the building block for arbitrary high order Bose-Einstein correlation functions. We assume for

simplicity that the core is fully incoherent.

Such a pattern of particle production is visualized in Fig. 2.

If the production of particles with a given rapidity  $y$  is limited to a narrow region in space-time around  $\bar{\eta}$  and  $\bar{\tau}$ , and the sizes of the effective source are sufficiently small (if the Bose-Einstein correlation function is sufficiently broad), the exponent of the  $\exp(iq \cdot \Delta x)$  factor of the Fourier-transformation can be decomposed in the shaded region in Fig. 2 as

$$q_0 t - q_z r_z \simeq Q_=(\tau - \bar{\tau}) - Q_{\parallel} \bar{\tau}(\eta - \bar{\eta}), \quad (22)$$

$$q_x r_x + q_y r_y \equiv Q_{\parallel} r_{\parallel} + Q_{\perp} r_{\perp}. \quad (23)$$

The invariant *temporal*, *parallel*, *sideward*, *outward* (and *perpendicular*) relative momentum components are defined, respectively, as

$$Q_ = q_0 \cosh[\bar{\eta}] - q_z \sinh[\bar{\eta}], \quad (24)$$

$$Q_{\parallel} = q_z \cosh[\bar{\eta}] - q_0 \sinh[\bar{\eta}], \quad (25)$$

$$Q_{\perp} = (q_x K_y - q_y K_x) / \sqrt{K_x^2 + K_y^2}, \quad (26)$$

$$Q_{\parallel} = (q_x K_x + q_y K_y) / \sqrt{K_x^2 + K_y^2}, \quad (27)$$

$$Q_{\perp} = \sqrt{q_x^2 + q_y^2} = \sqrt{Q_{\parallel}^2 + Q_{\perp}^2}. \quad (28)$$

The timelike normal-vector  $\bar{n}$  indicates an invariant direction of the source in coordinate space [18]. It is parameterized as  $\bar{n}^{\mu} = (\cosh[\bar{\eta}], 0, 0, \sinh[\bar{\eta}])$ , where  $\bar{\eta}$  is a mean space-time rapidity [18,21,19]. The parameter  $\bar{\eta}$  is one of the fitted parameters in the BL type of decomposition of the relative momenta. The above equations are invariant, they can be evaluated in any frame. To simplify the presentation, in the following we evaluate  $q$  and  $\bar{\eta}$  in the LCMS. The acronym LCMS stands for the Longitudinal Center of Mass System, where the mean momentum of a particle pair has vanishing longitudinal component,  $K_z = 0.5(k_{1,z} + k_{2,z}) = 0$ . In this frame, introduced in ref. [22],  $\mathbf{K}$  is orthogonal to the beam axis, and the time-like information on the duration of the particle emission couples to the out direction. The rapidity of the LCMS frame can be easily found from the measurement of the momentum vectors of the particles. As  $\bar{\eta}$  is from now on a space-time rapidity measured in

the LCMS frame, it is invariant to longitudinal boosts:  $\bar{\eta}' = (\bar{\eta} - y) - (0 - y) = \bar{\eta}$ .

The symbolic notation for the *side* direction is two dots side by side as in  $Q_{\perp\perp}$ . The remaining transverse direction, the *out* direction was indexed as in  $Q_{\parallel}$ , in an attempt to help to distinguish the zero-th component of the relative momentum  $Q_0$  from the out component of the relative momentum  $Q_{\parallel} \equiv Q_o = Q_{out}$ ,  $Q_0 \neq Q_o$ . Hence  $K_{\perp} = |\mathbf{K}_{\perp}|$  and  $K_{\parallel} = 0$ . The geometrical idea behind this notation is explained in details in ref. [21]. The perpendicular (or transverse) component of the relative momentum is denoted by  $Q_{\perp}$ . By definition,  $Q_{\perp}$ ,  $Q_{\parallel}$  and  $Q_{\perp\perp}$  are invariants to longitudinal boosts, and  $Q^2 = -q \cdot q = Q_{\perp}^2 + Q_{\parallel}^2 + Q_{\perp\perp}^2 - Q_{\perp}^2$ .

A further simplification is obtained if we assume that the emission (or Wigner) function factorizes as a product of an *effective* proper-time distribution, a space-time rapidity distribution and a transverse coordinate distribution [22,18]:

$$S_c(x, K) d^4 x = H_*(\tau) G_*(\eta) I_*(r_x, r_y) \times d\tau \bar{\tau} d\eta dr_x dr_y. \quad (29)$$

The subscript  $*$  stands for a dependence on the mean momentum  $K$ , the mid-rapidity  $y_0$  and the scale of violation of boost-invariance  $\Delta\eta$ , using the symbolic notation  $f_* \equiv f[K, y_0, \Delta\eta]$ . The function  $H_*(\tau)$  stands for such an effective proper-time distribution (that includes, by definition, an extra factor  $\tau$  from the Jacobian  $d^4 x = d\tau \tau d\eta, dr_x dr_y$ , in order to relate the two-particle Bose-Einstein correlation function to a Fourier-transformation of a distribution function in  $\tau$ ). The effective space-time rapidity distribution is denoted by  $G_*(\eta)$ , while the effective transverse distribution is denoted by  $I_*(r_x, r_y)$ . In eq. (29), the mean value of the proper-time  $\bar{\tau}$  is factored out, to keep the distribution functions dimensionless.

With the help of the *small source size* (or large relative momentum) expansion of eq. (22), the amplitude  $\tilde{s}_c^i(1, 2) \propto \langle a_1^\dagger a_2 \rangle$  that determines the arbitrary order Bose-Einstein correlation functions can be written as follows:

$$\tilde{s}_c^i(1, 2) = \frac{\tilde{H}_*(Q_ =) \tilde{G}_*(Q_{\parallel}) \tilde{I}_*(Q_{\perp}, Q_{\perp\perp})}{\tilde{H}_*(0) \tilde{G}_*(0) \tilde{I}_*(0, 0)}. \quad (30)$$

The Fourier-transformed distributions read as

$$\tilde{H}_*(Q_+) = \int d\tau e^{iQ_+\tau} H_*(\tau), \quad (31)$$

$$\tilde{G}_*(Q_{||}) = \int d\eta e^{-iQ_{||}\eta} G_*(\eta), \quad (32)$$

$$\tilde{I}_*(Q_+, Q_{||}) = \int dr_+ dr_{||} e^{-iQ_+ r_+ - iQ_{||} r_{||}} I_*(r_+, r_{||}). \quad (33)$$

Utilizing the core-halo picture [5,20], the two-particle BECF can be written into a factorized Buda-Lund form as

$$C(\mathbf{k}_1, \mathbf{k}_2) = 1 + \lambda_* \frac{|\tilde{H}_*(Q_+)|^2}{|\tilde{H}_*(0)|^2} \frac{|\tilde{G}_*(Q_{||})|^2}{|\tilde{G}_*(0)|^2} \frac{|\tilde{I}_*(Q_+, Q_{||})|^2}{|\tilde{I}_*(0,0)|^2}. \quad (34)$$

Thus, the BL results are rather generic. For example, BL parameterization may in particular limiting cases yield the *power-law*, the *exponential*, the *double-Gaussian*, the *Gaussian*, or the less familiar *oscillating* forms of ref. [21]. The *Edgeworth*, the *Laguerre* or other similarly constructed low-momentum expansions [23] can be applied to any of the factors of one variable in eq. (34) to characterize these unknown shapes in a really model-independent manner, relying only on the convergence properties of expansions in terms of complete orthonormal sets of functions [23], as discussed below.

In a Gaussian approximation and assuming that  $R_+ = R_{||} = R_{\perp}$ , the Buda-Lund form of BECF reads as follows:

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = 1 + \lambda_* e^{-R_+^2 Q_+^2 - R_{||}^2 Q_{||}^2 - R_{\perp}^2 Q_{\perp}^2}, \quad (35)$$

where the 5 fit parameters are  $\lambda_*$ ,  $R_+$ ,  $R_{||}$ ,  $R_{\perp}$  and the value of  $\bar{\eta}$  that enters the definitions of  $Q_+$  and  $Q_{||}$  in eqs. (24,25). The fit parameter  $R_+$  reads as  $R$ -timelike, and this variable measures a width of the proper-time distribution  $H_*$ . The fit parameter  $R_{||}$  reads as  $R$ -parallel, it measures an invariant length parallel to the direction of the expansion. The fit parameter  $R_{\perp}$  reads as  $R$ -perpendicular or  $R$ -perp. For cylindrically symmetric sources,  $R_{\perp}$  measures a transversal rms radius of the particle emitting source.

The BL radius parameters characterize the lengths of homogeneity [24] in a longitudinally boost-invariant manner. The lengths

of homogeneity are generally smaller than the momentum-integrated, total extension of the source, they measure a region in space and time, where particle pairs with a given mean momentum  $\mathbf{K}$  are emitted from.

In eq. (35), the spatial information about the source distribution in  $(r_x, r_y)$  was combined to a single perp radius parameter  $R_{\perp}$ . In a more general Gaussian form, suitable for studying rings of fire and opacity effects, the Buda-Lund invariant BECF can be denoted as

$$C_2(q, K) = 1 + \lambda_* e^{-R_+^2 Q_+^2 - R_{||}^2 Q_{||}^2 - R_{\perp}^2 Q_{\perp}^2}. \quad (36)$$

The 6 fit parameters are  $\lambda_*$ ,  $R_+$ ,  $R_{||}$ ,  $R_{\perp}$ ,  $R_+$  and  $\bar{\eta}$ , all are in principle functions of  $(\mathbf{K}, y_0, \Delta\eta)$ . Note, that this equation is identical to eq. (44) of ref. [18], rewritten into the new, symbolic notation of the Lorentz-invariant directional decomposition.

The above equation may be relevant for a study of expanding shells, or rings of fire, as discussed first in ref. [18]. We argued, based on a simultaneous analysis of particle spectra and correlations, and on recently found exact solutions of non-relativistic fireball hydrodynamics [25] that an expanding, spherical shell of fire is formed by the protons in 30 A MeV  $^{40}\text{Ar} + ^{197}\text{Au}$  reactions, and that a two-dimensional, expanding ring of fire is formed in the transverse plane in NA22  $h + p$  reactions at CERN SPS [5].

Opacity effects, as suggested recently by H. Heiselberg [26], also require the distinction between  $R_+$  and  $R_{\perp}$ . The lack of transparency in the source may result in an effective source function, that looks like a crescent in the side-out reference frame [26]. When integrated over the direction of the mean momentum, the effective source looks like a ring of fire in the  $(r_x, r_y)$  frame.

The price of the invariant decomposition of the basic building blocks of any order Bose-Einstein correlation functions in the BL parameterization is that the correlation functions cannot be directly binned in the BL variables, as these can be determined after the parameter  $\bar{\eta}$  is fitted to the data – so the correlation function has to be binned first in some directly measurable relative momentum components, e.g. the (side,out,long) relative momenta in the LCMS frame. After fitting  $\bar{\eta}$  in

an arbitrary frame, the BECF can be rebinned into the BL form.

Other, more conventional parameterizations of the two-particle Bose-Einstein correlation functions are known as the Bertsch-Pratt (BP) and the Yano-Koonin-Podgoretskii (YKP) parameterizations. These parameterizations exist only in Gaussian forms for the two-particle BECF, while BL forms exist for non-Gaussian Bose-Einstein correlations of arbitrary number of bosons. Other advantages and drawbacks of the BP and the YKP forms as compared to the Gaussian version of BL were discussed in detail in refs. [21,5].

#### 4. Model-independent analysis of short-range correlations

The invariant Buda-Lund form corresponds to a small source size, “large” relative momentum expansion for the basic building block of the Bose-Einstein correlation functions. So it is natural to apply an expansion technique that is based on correction terms at small relative momentum, corresponding to large source sizes. This will be reviewed below following the lines of refs. [23,5].

The reviewed method is *really* model-independent, and it can be applied not only to Bose-Einstein correlation functions but to every experimentally determined function, which features the properties *i)* and *ii)* listed below.

The following experimental properties are assumed:

- i)* The measured function tends to a constant for large values of the relative momentum.
- ii)* The measured function has a non-trivial structure at a certain value of its argument.

The location of the non-trivial structure in the correlation function is assumed for simplicity to be close to  $Q = 0$ .

The properties *i)* and *ii)* are well satisfied by e.g. the conventionally used two-particle Bose-Einstein correlation functions. For a critical review on the non-ideal features of short-range correlations, (e.g. non-Gaussian shapes in multi-dimensional Bose-Einstein correlation studies), we recommend ref. [11].

The core/halo intercept parameter  $\lambda_*$  is defined

as the *extrapolated* value of the two-particle correlation function at  $Q = 0$ , see ref [5] for greater details. It turns out, that  $\lambda_*$  is an important physical observable, related to the degree of partial restoration of  $U_A(1)$  symmetry in hot and dense hadronic matter [27,28].

A really model-independent approach is to expand the measured correlation functions in an abstract Hilbert space of functions. It is reasonable to formulate such an expansion so that already the first term in the series be as close to the measured data points as possible. This can be achieved if one identifies [23,29] the approximate shape (e.g. the approximate Gaussian or the exponential shape) of the correlation function with the abstract measure  $\mu(t)dt$  in the abstract Hilbert-space  $\mathcal{H}$ . The orthogonality of the basis functions  $\phi_n(t)$  in  $\mathcal{H}$  can be utilized to guarantee the convergence of these kind of expansions, see refs. [23,29] for greater details.

##### 4.1. Laguerre expansion

If in a zeroth order approximation the correlation function has an exponential shape, then it is an efficient method to apply the Laguerre expansion, as a special case of the general formulation of ref. [23,29]:

$$C_2(Q) = \mathcal{N} \left\{ 1 + \lambda_L \exp(-QR_L) \times \left[ 1 + c_1 L_1(QR_L) + \frac{c_2}{2!} L_2(QR_L) + \dots \right] \right\}. \quad (37)$$

In this and the next subsection,  $Q$  stands symbolically for any, experimentally chosen, one dimensional relative momentum variable. The fit parameters are the scale parameters  $\mathcal{N}$ ,  $\lambda_L$ ,  $R_L$  and the expansion coefficients  $c_1$ ,  $c_2$ , ... . The order  $n$  Laguerre polynomials are defined as

$$L_n(t) = \exp(t) \frac{d^n}{dt^n} t^n \exp(-t), \quad (38)$$

they form a complete orthogonal basis for an exponential measure as

$$\int_0^\infty dt \exp(-t) L_n(t) L_m(t) \propto \delta_{n,m}. \quad (39)$$

The first few Laguerre polynomials are explicitly given as

$$L_0(t) = 1, \quad (40)$$

$$L_1(t) = t - 1, \quad (41)$$

$$L_2(t) = t^2 - 4t + 2, \dots \quad (42)$$

As the Laguerre polynomials are non-vanishing at the origin,  $C(Q = 0) \neq 1 + \lambda_L$ . The physically significant core/halo intercept parameter  $\lambda_*$  can be obtained from the parameter  $\lambda_L$  of the Laguerre expansion as

$$\lambda_* = \lambda_L[1 - c_1 + c_2 - \dots]. \quad (43)$$

#### 4.2. Edgeworth expansion

If, in a zeroth-order approximation, the correlation function has a Gaussian shape, then the general form given in ref. [30] takes the particular form of the Edgeworth expansion [29–31] as:

$$C(Q) = \mathcal{N} \left\{ 1 + \lambda_E \exp(-Q^2 R_E^2) \times \left[ 1 + \frac{\kappa_3}{3!} H_3(\sqrt{2} Q R_E) + \frac{\kappa_4}{4!} H_4(\sqrt{2} Q R_E) \dots \right] \right\} \quad (44)$$

The fit parameters are the scale parameters  $\mathcal{N}$ ,  $\lambda_E$ ,  $R_E$  and the expansion coefficients  $\kappa_3$ ,  $\kappa_4$ , ..., that coincide with the cumulants of rank 3, 4, ..., of the correlation function. The Hermite polynomials are defined as

$$H_n(t) = \exp(t^2/2) \left( -\frac{d}{dt} \right)^n \exp(-t^2/2), \quad (45)$$

they form a complete orthogonal basis for an Gaussian measure as

$$\int_{-\infty}^{\infty} dt \exp(-t^2/2) H_n(t) H_m(t) \propto \delta_{n,m}. \quad (46)$$

The first few Hermite polynomials are listed as

$$H_1(t) = t, \quad (47)$$

$$H_2(t) = t^2 - 1, \quad (48)$$

$$H_3(t) = t^3 - 3t, \quad (49)$$

$$H_4(t) = t^4 - 6t^2 + 3, \dots \quad (50)$$

The physically significant core/halo intercept parameter  $\lambda_*$  can be obtained from the Edgeworth fit of eq. (44) as

$$\lambda_* = \lambda_E \left[ 1 + \frac{\kappa_4}{8} + \dots \right]. \quad (51)$$

This expansion technique was applied in the conference contributions [29,30] to the AFS minimum bias and 2-jet events to characterize successfully the deviation of data from a Gaussian

shape. It was also successfully applied to characterize the non-Gaussian nature of the correlation function in two-dimensions in case of the preliminary E802 data in ref. [29], and it was recently applied to characterize the non-Gaussian nature of the three-dimensional two-pion BECF in  $e^+ + e^-$  reactions at LEP1 [32].

Fig. 1 of ref. [23] indicates the ability of the Laguerre expansions to characterize two well-known, non-Gaussian correlation functions [23]: the second-order short-range correlation function  $D_2^s(Q)$  as determined by the UA1 and the NA22 experiments [33,34]. The fit results were summarized in Table 1 of ref. [23].

From the fit results, the core/halo model intercept parameter is obtained as  $\lambda_* = 1.14 \pm 0.10$  (UA1) and  $\lambda_* = 1.11 \pm 0.17$  (NA22). As both of these values are within errors equal to unity, the maximum of the possible value of the intercept parameter  $\lambda_*$  in a fully chaotic source, we concluded [23] that either there are other than Bose-Einstein short-range correlations observed by both collaboration, or in case of this measurement the full halo of long lived resonances is resolved [20,35–37].

If the two-particle BECF can be factorized as a product of (two or more) functions of one variable each, then the Laguerre and the Edgeworth expansions can be applied to the multiplicative factors – functions of one variable, each, as done in refs. [32,23,29,30]. The full, non-factorized form of two-dimensional Edgeworth expansion and the interpretation of its parameters is described in the handbook on mathematical statistics by Kendall and Stuart [38].

#### 5. Binary source formalism

The first experimental evidence for oscillating behaviour in the two-particle correlation function has been observed by the NA49 collaboration, unexpectedly, in the proton-proton correlations in central Pb+Pb collisions at CERN SPS [39]. The frequency of the oscillations has been explained with the help of the binary structure of the proton source in ref. [5]. Here we follow the lines of this presentation to explain the relationship between binary sources and oscillations in two-particle in-



terferometry.

Binary sources appear generally: in astrophysics, in form of binary stars, in particle physics, in form of  $W^+W^-$  pairs, that separate before they decay to hadrons.

Let us consider first the simplest possible example, to see how the binary sources result in oscillations in the Bose-Einstein or Fermi-Dirac correlation function. Suppose a source distribution  $s(x - x_+)$  describes, for example, a Gaussian source centered on  $x_+$ . Consider a binary system, where the emission happens from  $s_+ = s(x - x_+)$  with fraction  $f_+$ , or from a displaced source,  $s_- = s(x - x_-)$ , centered on  $x_-$ , with a fraction  $f_-$ . For such a binary source, the amplitude of the emission is

$$\rho(x) = f_+ s(x - x_+) + f_- s(x - x_-), \quad (52)$$

and the normalization requires

$$f_+ + f_- = 1 \quad (53)$$

The two-particle Bose-Einstein or Fermi-Dirac correlation function is

$$C(q) = 1 \pm |\tilde{\rho}(q)|^2 = 1 \pm \Omega(q) |\tilde{s}(q)|^2, \quad (54)$$

where  $+$  is for bosons, and  $-$  for fermions. The oscillating pre-factor  $\Omega(q)$  satisfies  $0 \leq \Omega(q) \leq 1$  and  $\Omega(0) = 1$ . This factor is given as

$$\Omega(q) = [(f_+^2 + f_-^2) + 2f_+f_- \cos[q(x_+ - x_-)]] \quad (55)$$

The strength of the oscillations is controlled by the relative strength of emission from the displaced sources and the period of the oscillations can be used to learn about the distance of the emitters. In the limit of one emitter ( $f_+ = 1$  and  $f_- = 0$ , or vice versa), the oscillations disappear.

In particle physics, the effective separation between the sources can be estimated from the uncertainty relation to be  $x_{\pm} = |x_+ - x_-| \approx 2\hbar/M_W \approx 0.005$  fm. Although this is much smaller, than the effective size of the pion source, 1 fm, one has to keep in mind that the back-to-back momenta of the  $W^+W^-$  pairs can be large, as compared to the pion mass. Due to this boost, pions with similar momentum may be emitted from different  $W$ -s with a separation which is already comparable to the 1 fm hadronization scale,

and the resulting oscillations may become observable.

In stellar astronomy, the separation between the binary stars is typically much larger than the diameter of the stars, hence the oscillations are well measurable. In principle, similar oscillations may provide a tool to measure the separation of the  $W^+$  from  $W^-$  in 4-jet events at LEP2. The scale of separation of  $W^+W^-$  pairs is a key observable to estimate in a quantum-mechanically correct manner the influence of the Bose-Einstein correlations on the reconstruction of the  $W$  mass.

## 6. Oscillations in two-proton correlations

In heavy ion physics, oscillations are seen in the short range of the  $p + p$  correlation function [39], with a half-period of  $Q_h = 30$  MeV, in Pb+Pb reactions at CERN SPS measured by the NA49 collaboration. Proton-proton short-range correlations are strongly influenced not only by the Fermi-Dirac statistics but also by Coulomb and strong final state interactions. This implies that the proton-proton correlation function of a binary proton source is approximately given by

$$C^{pp}(q) \simeq G(q)[1 + \Omega(q)(C_s^{0,pp}(q) - 1)] \quad (56)$$

where  $C_s^{0,pp}(q)$  stands for the two-proton correlation function including only the effects of strong final state interactions for one component of the binary source,  $s(x)$ . This function  $C_s^{0,pp}(q)$  can be evaluated similarly to the calculations performed for the conventional, one component, Gaussian like sources, see e.g. ref. [41], with the help of the proton-proton relative wave-function  $\phi_{\mathbf{q}}^0(\mathbf{x})$  which deviates from a relative plane wave due to the strong final state interactions. In eq. (56), the binary nature of the source is represented by the oscillating pre-factor  $\Omega(q)$  the Coulomb final state interactions are represented in a small source size approximation by a conventional Gamow penetration factor  $G(q)$ . The above approximation assumes that the strong final state interactions act on a scale that is smaller than the scale of separation of the binary proton source, hence the form of the oscillating pre-factor  $\Omega(q)$  is given by eq. (55). The 30 MeV half-period of the oscillations in NA49 data implies a separation of

$x_{\pm} = \pi\hbar/Q_h \approx 20$  fm, which is indeed much larger than the typical range of strong final state interactions. Hence the oscillations in  $C_2^{pp}$  can be attributed to interference between the two peaks of the NA49 proton  $dn/dy$  distribution [42], separated by  $\Delta y = 2.5$ . As for the protons we have  $m \gg T_0 = 140$  MeV, we can identify this rapidity difference with the space-time rapidity difference between the two peaks of the rapidity distribution. The longitudinal scale of the separation is then given by  $x_{\pm} = 2\bar{\tau} \sinh(\Delta\eta_p/2)$ , which can be used to estimate the mean freeze-out time of protons,  $\bar{\tau} = \pi\hbar/[2Q_h \sinh(\Delta\eta_p/2)] \approx 6.4$  fm/c, in a good agreement with the average value of  $\bar{\tau} = 5.9 \pm 0.6$  as extracted from the simultaneous analysis of the single-particle spectra and HBT radii in NA44, NA49 and WA98 experiments in the Buda-Lund picture, as summarized in refs. [43,5].

Note that the  ${}^2He$  resonance is responsible for magnifying the structure of the two-proton correlation function at  $Q \approx 50 - 100$  MeV. This magnification makes the oscillations visible, in spite of their relatively large half period. Another fortunate development in Pb+Pb collisions was that the  $dn/dy$  distribution of the net baryon number (protons) developed two separate peaks, indicating perhaps the onset of nuclear transparency at 160 AGeV. At the smaller AGS bombarding energies, the net baryon number has a single maximum at mid-rapidity which implies a very small, almost vanishing separation of the proton source into two effective components, that correspond to unobservably large period of oscillations,  $\Omega \approx 1$  within the resolvable relative momentum range.

## 7. Oscillating Bose-Einstein correlations

The two-particle Bose-Einstein correlation function of the Buda-Lund hydro model (BL-H) was evaluated in ref.[18] in a Gaussian approximation, without applying the binary source formulation. An improved calculation was presented in ref. [21], where the correlation function was evaluated using in the binary source formulation, and the corresponding oscillations were found.

Using the exponential form of the  $\cosh[\eta - y]$  factor, the BL-H emission function  $S_c(x, \mathbf{k})$  can

be written as a sum of two terms:

$$S_c(x, \mathbf{k}) = 0.5[S_+(x, \mathbf{k}) + S_-(x, \mathbf{k})], \quad (57)$$

$$S_{\pm}(x, \mathbf{k}) = \frac{g}{(2\pi)^3} \frac{m_t \exp[\pm\eta \mp y] H_*(\tau)}{[f_B(x, \mathbf{k}) + s]}, \quad (58)$$

$$f_B(x, \mathbf{k}) = \exp\left[\frac{k^\mu u_\mu(x) - \mu(x)}{T(x)}\right], \quad (59)$$

and  $s = 0, \pm 1$  for Boltzmann, Fermi-Dirac or Bose-Einstein distributions.

The above splitting is the basis of the binary source formulation of the BL-H parameterization. The effective emission function components are both subject to Fourier - transformation in the BL approach. In an improved saddle-point approximation, the two components  $S_+(x, k)$  and  $S_-(x, k)$  can be Fourier - transformed independently, finding the separate maxima (saddle point)  $\bar{x}_+$  and  $\bar{x}_-$  of  $S_+(x, k)$  and  $S_-(x, k)$ , and performing the analytic calculation for the two components separately.

These oscillations in the intensity correlation function are similar to the oscillations in the intensity correlations of photons from binary stars in stellar astronomy [44].

Note that the oscillations are expected to be small in the BL-H picture, and the Gaussian remains a good approximation to the BECF but with modified radius parameters.

The Buda-Lund hydro parameterization (BL-H) was invented in the same paper as the BL parameterization of the Bose-Einstein correlation functions [18], but in principle the general BL forms of the correlation function do not depend on the hydrodynamical ansatz (BL-H).

The BL-H two-particle Bose-Einstein correlation function is evaluated in the binary source formalism in ref. [21]:

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = 1 + \lambda_* \Omega e^{-Q_{\parallel}^2 \bar{R}_{\parallel}^2 - Q_{\perp}^2 \bar{R}_{\perp}^2 - Q_{\perp}^2 \bar{R}_{\perp}^2}, \quad (60)$$

where the pre-factor  $\Omega$  induces oscillations within the Gaussian envelope as a function of  $Q_{\parallel}$ . This oscillating pre-factor satisfies  $0 \leq \Omega(Q_{\parallel}) \leq 1$  and  $\Omega(0) = 1$ . This factor is given as

$$\Omega = \cos^2(Q_{\parallel} \bar{R}_{\parallel} \Delta\bar{\eta}) \times [1 + \tan^2(Q_{\parallel} \bar{R}_{\parallel} \Delta\bar{\eta}) \tanh^2 \bar{\eta}]. \quad (61)$$

The invariant BL decomposition of the relative momentum is utilized to present the correlation function in the simplest possible form. Although the shape of the BECF is non-Gaussian, because the factor  $\Omega(Q_{\parallel})$  results in oscillations of the correlator, the result is still explicitly boost-invariant. Although the source is assumed to be cylindrically symmetric, we have 6 free fit parameters in this BL form of the correlation function:  $\lambda_*$ ,  $R_{=}$ ,  $R_{\parallel}$ ,  $R_{\perp}$ ,  $\bar{\eta}$  and  $\Delta\bar{\eta}$ . The latter two control the period of the oscillations in the correlation function, which in turn carries information on the separation of the effective binary sources. This emphasizes the importance of the oscillating factor in the BL Bose-Einstein correlation function.

Due to the analytically found oscillations, the presented form of the BECF goes beyond the single Gaussian version of the saddle-point calculations of ref. [45,46]. This result goes also beyond the results obtainable in the YKP or the BP parameterizations. In principle, the binary-source saddle-point calculation gives more accurate analytic results than the numerical evaluation of space-time variances, as the binary-source calculation keeps non-Gaussian information on the detailed shape of the Bose-Einstein correlation function. The parameters of the spectrum and the correlation function are the same, given in more details in ref. [5]. Hence the simultaneous analysis of the two-particle correlations and the single-particle spectra, advocated already in refs. [18], yields a rather precise picture of particle production. Without the oscillations, it is possible to determine only the means and the variances of the density, flow and temperature profiles of the particle source. By taking into account the oscillations, additional information about the separation of the source into an effective binary structure can also be established.

## 8. Summary and outlook

One of the new directions in two-particle quantum statistical correlation studies is to search for non-Gaussian structures. An important observation is that particle interferometry for binary sources predicts the existence of oscillations in the

two-particle Bose-Einstein or Fermi-Dirac correlation functions. Intensity correlations oscillate, if the sources are binary, and the separation of the binary sources is related to the period of the oscillations. Novel model-independent tools to search for such oscillatory patterns are the Edgeworth or the Laguerre expansion techniques. The frequency of oscillations in the two-proton correlation function in Pb + Pb collisions at CERN SPS has been explained in terms of the BL-H model as a consequence of the separation of the proton sources in this reaction.

More work is required to work out the greater details of oscillatory patterns in two-particle Bose-Einstein correlations.

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